

# Matter sound waves in two-component Bose-Einstein condensates

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## **Abstract.**

The creation and propagation of sound waves in two-component Bose-Einstein condensates (BEC) are investigated and a new method of wave generation in binary BEC mixtures is proposed. The method is based on a fast change of the inter-species interaction constant and is illustrated for two experimental settings: a drop-like condensate immersed into a second large repulsive condensate, and a binary mixture of two homogeneous repulsive BEC's. A mathematical model based on the linearized coupled Gross-Pitaevskii equations is developed and explicit formulae for the space and time dependence of sound waves are provided. Comparison of the analytical and numerical results shows excellent agreement, confirming the validity of the proposed approach.

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## 1. Introduction

Wave phenomena in Bose-Einstein condensates (BEC) represent a very active field of research since the first experimental realization of BEC. Matter wave solitons, shock waves and different wave instabilities are among the hot topics of the field presently under investigation (for a recent review see [1]). In particular, dispersive shock waves in BECs consisting of modulated nonlinear periodic waves which decompose into dark soliton trains (e.g., when an initial disturbance is represented by a hole in the density distribution) or eventually into small amplitude waves (e.g., when an initial disturbance is represented by a hump in the density distribution), have been recently investigated. Theory of such waves and soliton trains described by the nonlinear Schrödinger (NLS) equation was developed in [2, 3, 4] and applied to their formation in a one-component BEC in [5, 6, 7]. Shock waves in discrete NLS models were investigated in [8, 9] while experimental observations of shock waves in BEC were reported in [7, 10]. Although these studies were mainly concentrated on single component BEC, binary BEC mixtures, both single species (two distinct spin or hyperfine states of the same atom) [11] and double species [12], have been created and the existence of several kinds of nonlinear excitations in presence of optical lattices has been demonstrated [13].

In spite of the intrinsic nonlinearity of BECs and the relevance of nonlinear excitations for applications, small amplitude wave-packets associated with long-wavelength modulations of the condensate (matter sound waves) also appear to be important. These waves naturally arise in the linear or quasi-linear regimes, e.g. when the propagation of small amplitude disturbances on top of the condensate is considered or when the inter-particle interactions (nonlinearity) is “artificially” reduced by means of Feshbach resonances. They are generated also in a supersonic flow of BEC past an obstacle where they form so called “ship wave” patterns located outside the Mach cone [14, 15, 16, 17]. Small amplitude wave-packets should not be confused with dispersive shocks for which larger disturbances or larger nonlinearity are usually required (this is particularly true in the multi-dimensional case). From a mathematical point of view they can be characterized as waves which, opposite to solitons, entirely belong to the continuum spectrum of the linear eigenvalue problem associated with the NLS equation. From a physical point of view, these waves are composed of linear modes which have a Bogoliubov-like dispersion relation and which, in a pure quantum description, correspond to elementary excitations of the system. The spectrum of such excitations characterizes the properties of the two-component BEC superfluid [18].

Propagation of sound waves in single-component harmonically trapped BEC’s was experimentally observed in [10, 19] in the form of travelling density perturbations. Tightly focused far blue-detuned from atomic transitions laser beam applied to the condensate acts as expulsive potential and rapidly pushes atoms from the light field maximum. Depending on the laser power and pulse duration, different kinds of waves are created in the condensate. Smoothly propagating sound waves are created when the laser beam waist is narrow compared to the size of the condensate and the light intensity

is small enough. Sound waves in two-component BEC's were experimentally observed in [20]. While the advantages of experimental techniques for BECs involving laser beams are extremely appealing, there is a crucial drawback when it concerns applications to two component condensates (in Ref. [20] the slow light technique was used). The reason for this is connected to the fact that different atomic species feel different light-induced expulsive potentials (due to optical dipole force) and respond to the laser field in different manners. This leads to the appearance of interfering waves which complicate very much the process under investigation. From this point of view the search of alternative methods to generate matter waves in two component BECs is highly desirable.

The aim of this paper is twofold. From one side we investigate, both analytically and numerically, sound waves in two-component BEC's generated by the time evolution of a small initial disturbance. From the other side, we present a new method for the generation of different kinds of waves in binary BEC mixtures which is based on the fast change of the inter-species interaction constant. In view of the recent progress made in the manipulation of the inter-component interactions in binary BEC mixtures [21] by means of Feshbach resonances, the proposed method appears to be experimentally quite feasible and has the advantage that it relies entirely on the internal physical mechanism for the creation of the density disturbance. This method is used to investigate sound waves in two experimental settings: i) the case of a drop-like condensate immersed into a second large repulsive condensate and ii) the case of a homogeneous mixture of two repulsive BEC's both with finite backgrounds. In case i) we first create an initial disturbance in the form of a gray or a bright hump (depending on the sign of the interaction) on a finite background by adiabatically turning on the interaction between the drop-like condensate and the surrounding condensate, and then we rapidly remove the inter-species interaction. The energy is then entirely released in the form of sound waves in the background condensate. In case ii) the initial disturbance is created by a change of the intra- or inter-species interaction constant via optically induced Feshbach resonance. To illustrate our approach we numerically simulate the creation and evolution of matter sound waves in a two component BEC. A mathematical model based on the linearized Gross-Pitaevskii equation has been developed and explicit formulae for the space and time dependence of the sound waves are obtained. Comparison of the analytical and numerical results shows a very good agreement, confirming the validity of the proposed approach. The obtained results can be of interest for the physics of interacting two-component superfluid systems.

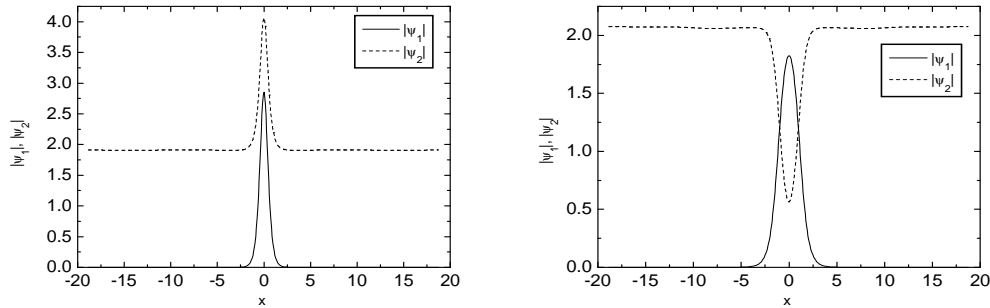
The outline of the paper is as follows. In Sec. II we illustrate the proposed approach for the one dimensional setting, by formulating the mathematical model and performing numerical simulations. In this case one component is a bright soliton or drop-like condensate immersed in the second large repulsive condensate with a finite background. Since the first component is on the zero background, after the interspecies interaction is turned off, we consider the waves generated in the second (background) condensate. Analytical formulae describing the space-time patterns of sound waves in coupled BECs are explicitly derived. In Sec. III the proposed method is extended to the

two dimensional case. In Sec. IV we consider the most interesting and experimentally relevant case of sound waves of two component repulsive condensates both with a finite background. In this case the inter-species interaction is present during the whole evolution of waves in both condensates, exhibiting interference effects in the dynamics. In Sec. V the main results are summarized.

## 2. One-dimensional condensates

### 2.1. Decay of a small disturbance of a stationary state to linear wave packets

Consider a quasi-1D two-component BEC with attractive interaction in one species and repulsive one in the other species. In absence of the interspecies interaction the stationary ground state corresponds to a bright soliton in the attractive condensate and the Thomas-Fermi (TF) distribution in the repulsive one (which can be considered as uniform if the TF radius is much greater than the bright soliton's width). Adiabatic turning on the interspecies interaction leads to the formation of a new stationary state with a hump or a hole in the repulsive BEC, depending on the sign of the interspecies interaction (see Fig. 1).



**Figure 1.** The stationary state of two coupled 1D BECs emerged from the attractive (left panel), and repulsive (right panel) inter-species interactions. Zero background condensate (solid line) is attractive, and finite background condensate (dashed line) is repulsive.

A fast turning off of the interspecies interaction will then create a disturbed stationary state in the repulsive condensate. In this section, we shall study evolution of such a perturbation. Since the attractive condensate has a zero background density, we can consider only waves created in the repulsive finite background condensate using the scalar GP equation.

In this case dynamics of the repulsive BEC is governed by the GP equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + (\mu - g|\psi|^2)\psi = 0, \quad (1)$$

where

$$\mu = g\rho_0 = c^2 \quad (2)$$

is the chemical potential,  $\rho_0$  is the density of an undisturbed uniform state, and  $c$  denotes the sound velocity of long wavelength waves propagation along such a uniform state. We imply here that the coupling constant  $g \equiv g_{1D}$  corresponds to the 1D geometry of the trap with strong radial confinement ( $\omega_\perp \gg \omega_x$ ) of the BEC.

Let the initial state be described by a disturbed wave function

$$\psi_0 = \sqrt{\rho_0} + \delta\psi_0, \quad |\delta\psi_0| \ll \sqrt{\rho_0}. \quad (3)$$

For example,  $\delta\psi_0$  can describe the hump or the hole created in the density distribution by the interaction with another condensate. Evolution of a small disturbance  $\delta\psi$  is governed by a linearized equation (1),

$$i\delta\psi_t + \frac{1}{2}\delta\psi_{xx} - c^2(\delta\psi + \delta\psi^*) = 0, \quad (4)$$

which should be solved with the initial condition

$$\delta\psi|_{t=0} = \delta\psi_0(x). \quad (5)$$

It is convenient to separate in this equation the real and imaginary parts by introducing

$$\delta\psi = A + iB, \quad (6)$$

so that it transforms to the system

$$\begin{aligned} A_t + \frac{1}{2}B_{xx} &= 0, \\ B_t - \frac{1}{2}A_{xx} + 2c^2A &= 0. \end{aligned} \quad (7)$$

The variable  $B$  can be easily excluded and we arrive at a single linear equation

$$A_{tt} - c^2A_{xx} + \frac{1}{4}A_{xxxx} = 0, \quad (8)$$

which can be readily solved by the Fourier method. To this end, we notice that linear harmonic waves  $A \propto \exp[i(kx - \omega t)]$  satisfy, as it should be, to the Bogoliubov dispersion law,

$$\omega = \pm\omega(k), \quad \omega(k) = k\sqrt{c^2 + \frac{k^2}{4}}. \quad (9)$$

Hence the general solution of Eq. (8) has the form

$$A(x, t) = \int_{-\infty}^{\infty} W_1(k) e^{i(kx - \omega(k)t)} \frac{dk}{2\pi} + \int_{-\infty}^{\infty} W_2(k) e^{i(kx + \omega(k)t)} \frac{dk}{2\pi}, \quad (10)$$

where  $W_{1,2}(k)$  are to be determined from the initial conditions:

$$\begin{aligned} A(x, 0) &\equiv A_0(x) = \int_{-\infty}^{\infty} [W_1(k) + W_2(k)] e^{ikx} \frac{dk}{2\pi}, \\ A_t(x, 0) &\equiv A_{t0}(x) = -i \int_{-\infty}^{\infty} \omega(k) [W_1(k) - W_2(k)] e^{ikx} \frac{dk}{2\pi}. \end{aligned} \quad (11)$$

The inverse Fourier transform gives the equations

$$A_0(k) = W_1(k) + W_2(k), \quad A_{t0}(k) = -i\omega(k)[W_1(k) - W_2(k)],$$

which readily provide

$$\begin{aligned} W_1(k) &= \frac{1}{2} \left( A_0(k) + \frac{i}{\omega(k)} A_{t0}(k) \right), \\ W_2(k) &= \frac{1}{2} \left( A_0(k) - \frac{i}{\omega(k)} A_{t0}(k) \right). \end{aligned} \quad (12)$$

The first equation (7) allows to express  $A_{t0}(k)$  in terms of  $B_0(k)$ ,  $A_{t0}(k) = (k^2/2)B_0(k)$ . At last,  $A(k)$  and  $B(k)$  can be related with disturbances of the density and the phase in BEC,  $\psi = \sqrt{\rho_0 + \delta\rho} \exp(i\delta\phi) \cong \sqrt{\rho_0} + \frac{\delta\rho}{2\sqrt{\rho_0}} + i\sqrt{\rho_0}\delta\phi$ , so that

$$A = \frac{\delta\rho}{2\sqrt{\rho_0}}, \quad B = \sqrt{\rho_0}\delta\phi. \quad (13)$$

Hence, Eqs. (12) take the form

$$\begin{aligned} W_1(k) &= \frac{1}{4\sqrt{\rho_0}} \left( \delta\rho_0(k) + i\frac{k^2\rho_0}{\omega(k)}\delta\phi_0(k) \right), \\ W_2(k) &= \frac{1}{4\sqrt{\rho_0}} \left( \delta\rho_0(k) - i\frac{k^2\rho_0}{\omega(k)}\delta\phi_0(k) \right). \end{aligned} \quad (14)$$

Let us consider, for the sake of definiteness, the case of an initial disturbance of density only,  $\delta\phi_0 = 0$ , so that  $W_1(k) = W_2(k) = \delta\rho_0(k)/(4\sqrt{\rho_0})$  and then

$$\begin{aligned} \delta\rho(x, t) &= 2\sqrt{\rho_0}A(x, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \delta\rho_0(k) \left[ e^{i(kx - \omega(k)t)} + e^{i(kx + \omega(k)t)} \right] dk \\ &= \frac{1}{\pi} \int_0^{\infty} \delta\rho_0(k) \cos(kx) \cos\left(tk\sqrt{c^2 + \frac{k^2}{4}}\right) dk. \end{aligned} \quad (15)$$

The integrals here can be estimated for large  $t$ , as usual, by the method of stationary phase. For  $x > 0$ , only the first exponent gives contribution to the integral,

$$\delta\rho(x, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \delta\rho_0(k) e^{itf(k)} dk, \quad x > 0, \quad (16)$$

where

$$f(k) = k \left( \frac{x}{t} - \sqrt{c^2 + \frac{k^2}{4}} \right). \quad (17)$$

The values of  $k$  at the points of stationary phase determined by the condition  $df/dk = 0$  are equal to ( $x > ct$ )

$$k = \pm k_0(x, t), \quad k_0(x, t) = \frac{c}{\sqrt{2}} \left[ \left( \frac{x}{ct} \right)^2 - 4 + \frac{x}{ct} \sqrt{\left( \frac{x}{ct} \right)^2 + 8} \right]^{1/2}, \quad (18)$$

and integration yields

$$\delta\rho(x, t) = \frac{\delta\rho_0(k)}{\sqrt{2\pi t \left| \frac{d^2 f}{dk^2} \right|_0}} \cos \left[ tf(k_0) - \frac{\pi}{4} \right] \quad (19)$$

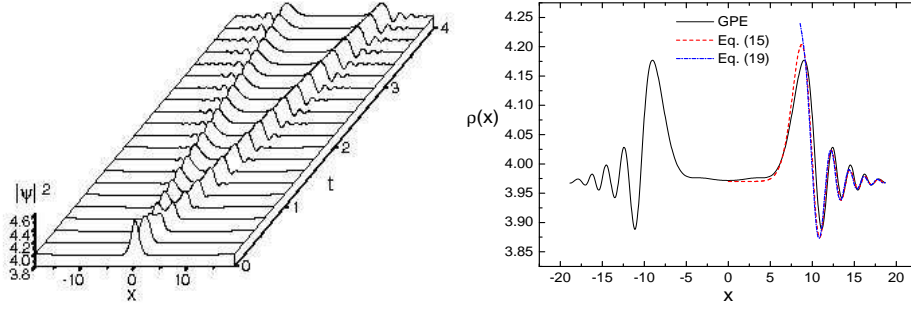
where

$$\left| \frac{d^2 f}{dk^2} \right|_0 = \frac{k_0(6c^2 + k_0^2)}{(4c^2 + k_0^2)^{3/2}}. \quad (20)$$

Here  $k_0$  is a function of  $x$  and  $t$  determined by Eq. (18). In Fig. 2 we compare the exact solution (15) corresponding to a gaussian initial disturbance

$$\delta\rho_0(x) = \frac{1}{\sqrt{\pi}a} \exp\left(-\frac{x^2}{a^2}\right), \quad \delta\rho_0(k) = \exp\left(-\frac{k^2 a^2}{4}\right), \quad (21)$$

with its approximation (19), and corroborate these two curves with numerical solution of the GP equation where we introduce as initial condition the wave profiles shown in the left panel of Fig. 1. As we see, the asymptotic formula (19) is quite accurate even for not very large values of  $t$  along almost all wave packet where  $|x| > ct$ .



**Figure 2.** Left panel: Time evolution of the density disturbance in the background BEC ( $\rho(x) = |\psi(x)|^2$ ), when the inter-species interaction constant is set to zero ( $g_{12} = 0$ ) at  $t = 0$ , according to GP equation with  $g_{22} = g = 1$ . Right panel: Comparison of the exact analytic solution (15), asymptotic approximation (19), and numerical solution of the GP equation for the density disturbance, corresponding to  $t = 4$  and  $c = 2$ . The initial disturbance has a gaussian form (21) with  $a = 0.95$ . Space dependence for analytical solutions is shown for  $x > 0$  at the moment  $t = 4$ . Similar calculation gives a symmetric distribution for  $x < 0$ .

### 3. Two-dimensional condensates

#### 3.1. Decay of small disturbances to wave packets

This problem can be solved by the same method which was used in 1D case with replacement  $\partial^2/\partial x^2 \rightarrow \Delta_\perp \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ . As a result we obtain the general solution in the form

$$A(\mathbf{r}, t) = \int W_1(\mathbf{k}) e^{i(\mathbf{k}\mathbf{r} - \omega(k)t)} \frac{d\mathbf{k}}{(2\pi)^2} + \int W_2(\mathbf{k}) e^{i(\mathbf{k}\mathbf{r} + \omega(k)t)} \frac{d\mathbf{k}}{(2\pi)^2}, \quad (22)$$

where  $W_{1,2}(\mathbf{k})$  are determined again from the initial conditions, which gives

$$\begin{aligned} W_1(\mathbf{k}) &= \frac{1}{4\sqrt{\rho_0}} \left( \delta\rho_0(\mathbf{k}) + i \frac{k^2 \rho_0}{\omega(k)} \delta\phi_0(\mathbf{k}) \right) \\ W_2(\mathbf{k}) &= \frac{1}{4\sqrt{\rho_0}} \left( \delta\rho_0(\mathbf{k}) - i \frac{k^2 \rho_0}{\omega(k)} \delta\phi_0(\mathbf{k}) \right). \end{aligned} \quad (23)$$

For cylindrically symmetric initial disturbance of density only ( $\delta\phi_0(\mathbf{k}) = 0$ ) we get for the wave of density  $\delta\rho(r, t) = 2\sqrt{\rho_0} A(r, t)$  the expressions similar to Eqs. (15),

$$\begin{aligned} \delta\rho(r, t) &= \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^\infty \delta\rho_0(k) \left[ e^{i(kr \cos \varphi - \omega(k)t)} + e^{i(kr \cos \varphi + \omega(k)t)} \right] k dk d\varphi, \\ &= \frac{1}{2\pi} \int_0^\infty \delta\rho_0(k) \cos \left( tk \sqrt{c^2 + \frac{k^2}{4}} \right) J_0(kr) k dk, \end{aligned} \quad (24)$$

where

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \cos \varphi} d\varphi \quad (25)$$

is the Bessel function of the first kind.

For  $kr \gg 1$  the integral in (24) can be estimated by the method of stationary phase with the use of asymptotic expression for the Bessel function,

$$J_0(kr) \approx \sqrt{\frac{2}{\pi kr}} \cos(kr - \pi/4). \quad (26)$$

As a result we get

$$\delta\rho(r, t) = \frac{\delta\rho_0(k)\sqrt{k_0}}{2\pi\sqrt{rt\left|\frac{d^2f}{dk^2}\right|_0}} \sin[tf(k_0)], \quad (27)$$

where

$$k_0(r, t) = \frac{c}{\sqrt{2}} \left[ \left( \frac{r}{ct} \right)^2 - 4 + \frac{r}{ct} \sqrt{\left( \frac{r}{ct} \right)^2 + 8} \right]^{1/2}, \quad r > ct, \quad (28)$$

$$f(k) = k \left( \frac{r}{t} - \sqrt{c^2 + \frac{k^2}{4}} \right), \quad (29)$$

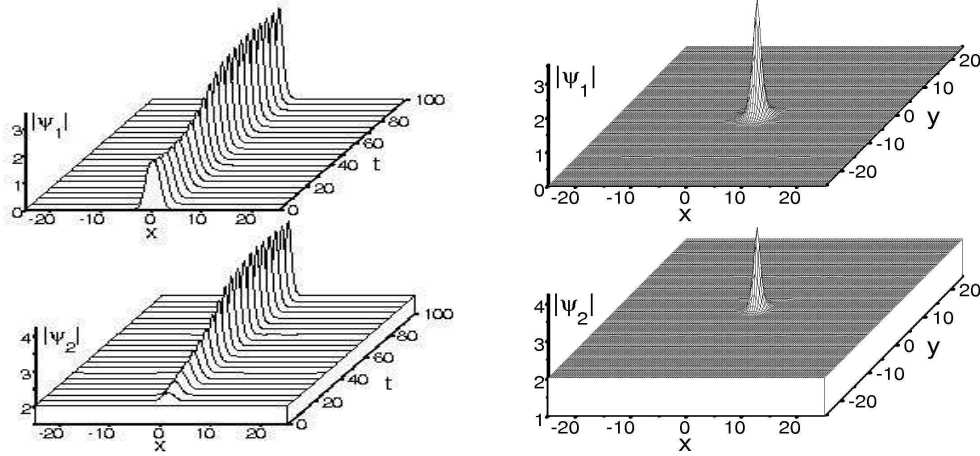
and  $|d^2f/dk^2|_0$  is given by (20).

In order to compare the analytical predictions with numerical solution of the GP equation we need to perform a sequence of simulations as in 1D case. However, there is an essential difference in the 2D case due to the fact that no stable localized solution for the drop-like condensate exists for the 2D GP equation without an external potential. To this regard we recall that the attractive 2D NLS equation admits a localized solution, the so called Townes soliton, which exist only for a critical value of the norm (number of atoms) and is unstable against small fluctuations of the critical norm (overcritical norms lead to collapse while undercritical ones lead to background decays). To avoid this problem, we prepare the drop-like condensate in the ground state of the parabolic trap  $V(x, y) = v_0(x^2 + y^2)$  with strength  $v_0 = 1$ , using the imaginary-time relaxation method [23]. Then we introduce the obtained ground state solution for the drop-like condensate  $\psi_1(x, y)$  and constant amplitude solution for the background condensate ( $\psi_2(x, y) = 2.0$ ) as initial condition in the coupled GP equation with  $g_{12} = 0$ , and slowly increase the inter-species interaction constant to a finite value  $g_{12} = -0.5$  (see Fig. 3 left panel). It is remarkable that after the localized coupled state has been created by this method, it remains stable even after the parabolic potential is removed, due to the stabilizing effect induced by the interspecies coupling (this state exists only due to coupling). As a result the initial state shown in the right panel of Fig. 3 is obtained. The density waves in the background condensate can be generated by fast turning off the interaction constant  $g_{12} \rightarrow 0$ . Decaying density hump induces circular wave in the background BEC traveling outward from the center. We have shown in Fig. 4 the dependence of  $\delta\rho$  on  $r$  at fixed value of time ( $t = 4$ ) for the gaussian initial distribution

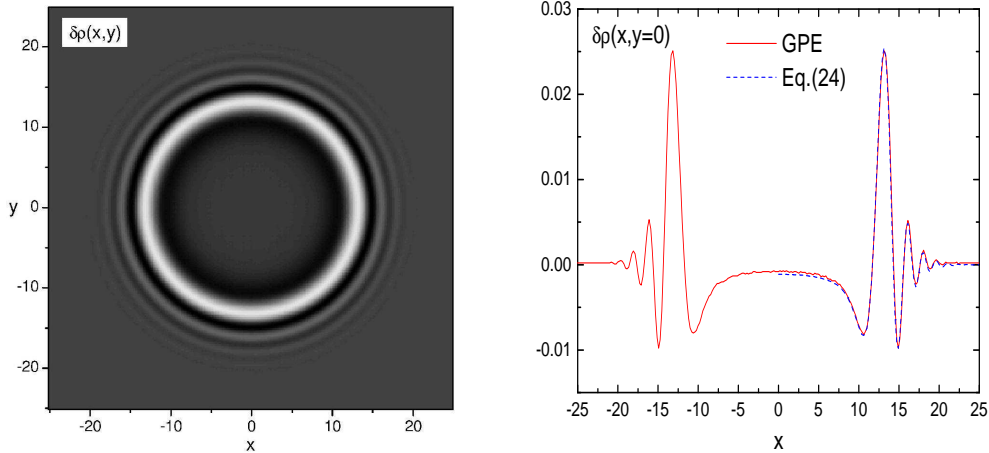
$$\delta\rho_0(r) = \frac{1}{\pi a^2} \exp\left(-\frac{r^2}{a^2}\right), \quad \delta\rho_0(k) = \exp\left(-\frac{k^2 a^2}{4}\right), \quad (30)$$

with  $a = 1$  calculated according to the exact expression (24). The asymptotic approximation (27) agrees very well with the exact expression (24) at  $x > ct$  (not shown in figure because the corresponding lines overlap).





**Figure 3.** Left panel: Formation of a density hump on the background condensate  $\psi_2(x, y)$ , shown at  $y = 0$  cross section, when the inter-species interaction constant is slowly raised from zero according to  $g_{12} = -0.5 \tanh(5t/t_{\text{end}})$ , with  $t_{\text{end}} = 100$ . Due to inter-species attraction, the drop-like repulsive condensate  $\psi_1(x, y)$  with a norm  $N = 6\pi$  becomes more narrow. Right panel: The final stationary coupled state of the two repulsive BEC's with  $g_{11} = 1$ ,  $g_{22} = 1$ , with attractive inter-species interaction constant  $g_{12} = -0.5$ .



**Figure 4.** Left panel: Density plot  $\rho(r) = |\psi(x, y)|^2$  of the repulsive background condensate at  $t = 4$  after turning off the inter-species interaction constant  $g_{12} = 0$ , according to numerical solution of the GP equation. White stripes correspond to higher density. Right panel: Comparison between analytical expression (24), for the density disturbance in the background condensate at  $t = 4$  with corresponding numerical solution of the GP equation ( $|\psi(x, y = 0)|^2 - \rho_0$ ), shown for  $y = 0$  cross section. Parameters:  $\rho_0 = 9$ ,  $c = 3$ ,  $a = 1.0$ .

#### 4. Two-component condensates

Let us now consider the relevant case of a homogeneous binary BEC mixture, when both components have a finite background. Two-component condensates are described

in our non-dimensional variables by the equations

$$\begin{aligned} i\frac{\partial\psi_1}{\partial t} + \frac{1}{2}\Delta\psi_1 + [\mu_1 - (g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2)]\psi_1 &= 0, \\ i\frac{\partial\psi_2}{\partial t} + \frac{1}{2}\Delta\psi_2 + [\mu_2 - (g_{12}|\psi_1|^2 + g_{22}|\psi_2|^2)]\psi_2 &= 0, \end{aligned} \quad (31)$$

where

$$\mu_1 = g_{11}\rho_{10} + g_{12}\rho_{20}, \quad \mu_2 = g_{12}\rho_{10} + g_{22}\rho_{20} \quad (32)$$

are chemical potentials of two species with equilibrium uniform densities  $\rho_{10}$  and  $\rho_{20}$ ; we keep here the general notation for the Laplacian  $\Delta$  which will be specified later depending on the number of spatial dimensions in the problem under consideration.

In disturbed condensates we have  $\psi_1 = \sqrt{\rho_{10}} + \delta\psi_1$ ,  $\psi_2 = \sqrt{\rho_{20}} + \delta\psi_2$ , where evolution of disturbances is governed by the linearized equations

$$\begin{aligned} i\delta\psi_1 + \frac{1}{2}\Delta\delta\psi_1 - g_{11}\rho_{10}(\delta\psi_1 + \delta\psi_1^*) - g_{12}\sqrt{\rho_{10}\rho_{20}}(\delta\psi_2 + \delta\psi_2^*) &= 0, \\ i\delta\psi_2 + \frac{1}{2}\Delta\delta\psi_2 - g_{12}\sqrt{\rho_{10}\rho_{20}}(\delta\psi_1 + \delta\psi_1^*) - g_{22}\rho_{20}(\delta\psi_2 + \delta\psi_2^*) &= 0. \end{aligned} \quad (33)$$

As before, we introduce

$$\delta\psi_1 = A_1 + iB_1, \quad \delta\psi_2 = A_2 + iB_2, \quad (34)$$

so that after substitution into (33), separation of real and imaginary parts followed by exclusion of  $B_1$  and  $B_2$  we arrive at the system

$$\begin{aligned} A_{1,tt} - g_{11}\rho_{10}\Delta A_1 + \frac{1}{4}\Delta^2 A_1 - g_{12}\sqrt{\rho_{10}\rho_{20}}\Delta A_2 &= 0, \\ A_{2,tt} - g_{22}\rho_{20}\Delta A_2 + \frac{1}{4}\Delta^2 A_2 - g_{12}\sqrt{\rho_{10}\rho_{20}}\Delta A_1 &= 0. \end{aligned} \quad (35)$$

For plane waves  $A_1, A_2 \propto \exp[i(\mathbf{k}\mathbf{r} - \omega t)]$  we reproduce the known dispersion laws

$$\omega_{\pm} = k\sqrt{c_{\pm}^2 + \frac{k^2}{4}}, \quad (36)$$

where the sound velocities corresponding to infinite wavelengths are equal to

$$c_{\pm} = \frac{1}{\sqrt{2}} \left[ g_{11}\rho_{10} + g_{22}\rho_{20} \pm \sqrt{(g_{11}\rho_{10} - g_{22}\rho_{20})^2 + 4g_{12}^2\rho_{10}\rho_{20}} \right]^{1/2}. \quad (37)$$

The ratios  $A_1/A_2$  in these two modes of linear waves are given by

$$\frac{A_1}{A_2} = \frac{2g_{12}\sqrt{\rho_{10}\rho_{20}}}{g_{22}\rho_{20} - g_{11}\rho_{10} \pm (c_+^2 - c_-^2)}. \quad (38)$$

Thus, the general solution of the system (35) is given by

$$\begin{aligned} A_1(\mathbf{r}, t) &= \int W_{11}^+(\mathbf{k})e^{i(\mathbf{k}\mathbf{r}-\omega_+t)}\frac{d^D k}{(2\pi)^D} + \int W_{12}^+(\mathbf{k})e^{i(\mathbf{k}\mathbf{r}+\omega_+t)}\frac{d^D k}{(2\pi)^D} \\ &+ \int W_{11}^-(\mathbf{k})e^{i(\mathbf{k}\mathbf{r}-\omega_-t)}\frac{d^D k}{(2\pi)^D} + \int W_{12}^-(\mathbf{k})e^{i(\mathbf{k}\mathbf{r}+\omega_-t)}\frac{d^D k}{(2\pi)^D}, \end{aligned} \quad (39)$$

$$\begin{aligned} A_2(\mathbf{r}, t) &= \int W_{21}^+(\mathbf{k})e^{i(\mathbf{k}\mathbf{r}-\omega_+t)}\frac{d^D k}{(2\pi)^D} + \int W_{22}^+(\mathbf{k})e^{i(\mathbf{k}\mathbf{r}+\omega_+t)}\frac{d^D k}{(2\pi)^D} \\ &+ \int W_{21}^-(\mathbf{k})e^{i(\mathbf{k}\mathbf{r}-\omega_-t)}\frac{d^D k}{(2\pi)^D} + \int W_{22}^-(\mathbf{k})e^{i(\mathbf{k}\mathbf{r}+\omega_-t)}\frac{d^D k}{(2\pi)^D}, \end{aligned} \quad (40)$$

where  $D$  denotes dimension of the space. According to (38) the Fourier components are related with each other by the equations

$$\frac{W_{11}^{\pm}}{W_{21}^{\pm}} = \frac{W_{12}^{\pm}}{W_{22}^{\pm}} = \frac{2g_{12}\sqrt{\rho_{10}\rho_{20}}}{g_{22}\rho_{20} - g_{11}\rho_{10} \pm (c_+^2 - c_-^2)}, \quad (41)$$

that is only four of them are independent of each other and should be determined from the initial conditions.

To simplify the notation, in what follows we shall consider the case when there exist only initial disturbances  $\delta\rho_{10}(\mathbf{r}), \delta\rho_{20}(\mathbf{r})$  of the densities. Then

$$W_{11}^{\pm} = W_{12}^{\pm} \equiv W_1^{\pm}, \quad W_{21}^{\pm} = W_{22}^{\pm} \equiv W_2^{\pm}, \quad (42)$$

and these four functions must satisfy the linear system

$$\begin{aligned} A_{10}(\mathbf{k}) &= 2(W_1^+ + W_1^-), \quad A_{20}(\mathbf{k}) = 2(W_2^+ + W_2^-), \\ \frac{W_1^+}{W_2^+} &= \frac{2g_{12}\sqrt{\rho_{10}\rho_{20}}}{g_{22}\rho_{20} - g_{11}\rho_{10} + (c_+^2 - c_-^2)}, \quad \frac{W_1^-}{W_2^-} = \frac{2g_{12}\sqrt{\rho_{10}\rho_{20}}}{g_{22}\rho_{20} - g_{11}\rho_{10} - (c_+^2 - c_-^2)}, \end{aligned} \quad (43)$$

which can be readily solved. Taking into account that

$$A_{10}(\mathbf{k}) = \frac{1}{2\sqrt{\rho_{10}}}\delta\rho_{10}(\mathbf{k}), \quad A_{20}(\mathbf{k}) = \frac{1}{2\sqrt{\rho_{20}}}\delta\rho_{20}(\mathbf{k}), \quad (44)$$

we can find the waves of the densities of the two species.

In 1D case the final formulae read

$$\begin{aligned} \delta\rho_1(x, t) &= \frac{1}{2\pi} \int_0^\infty \delta\rho_1(k) \cos kx \left[ \left( 1 + \frac{g_{11}\rho_{10} - g_{22}\rho_{20}}{c_+^2 - c_-^2} \right) \cos \left( tk\sqrt{c_+^2 + \frac{k^2}{4}} \right) \right. \\ &\quad \left. + \left( 1 - \frac{g_{11}\rho_{10} - g_{22}\rho_{20}}{c_+^2 - c_-^2} \right) \cos \left( tk\sqrt{c_-^2 + \frac{k^2}{4}} \right) \right] dk \\ &\quad + \frac{g_{12}\rho_{10}}{\pi(c_+^2 - c_-^2)} \int_0^\infty \delta\rho_2(k) \cos kx \left[ \cos \left( tk\sqrt{c_+^2 + \frac{k^2}{4}} \right) - \cos \left( tk\sqrt{c_-^2 + \frac{k^2}{4}} \right) \right] dk, \end{aligned} \quad (45)$$

$$\begin{aligned} \delta\rho_2(x, t) &= \frac{1}{2\pi} \int_0^\infty \delta\rho_2(k) \cos kx \left[ \left( 1 - \frac{g_{11}\rho_{10} - g_{22}\rho_{20}}{c_+^2 - c_-^2} \right) \cos \left( tk\sqrt{c_+^2 + \frac{k^2}{4}} \right) \right. \\ &\quad \left. + \left( 1 + \frac{g_{11}\rho_{10} - g_{22}\rho_{20}}{c_+^2 - c_-^2} \right) \cos \left( tk\sqrt{c_-^2 + \frac{k^2}{4}} \right) \right] dk \\ &\quad + \frac{g_{12}\rho_{20}}{\pi(c_+^2 - c_-^2)} \int_0^\infty \delta\rho_1(k) \cos kx \left[ \cos \left( tk\sqrt{c_+^2 + \frac{k^2}{4}} \right) - \cos \left( tk\sqrt{c_-^2 + \frac{k^2}{4}} \right) \right] dk. \end{aligned} \quad (46)$$

Similar formulae are obtained for two-dimensional cylindrically symmetric waves as:

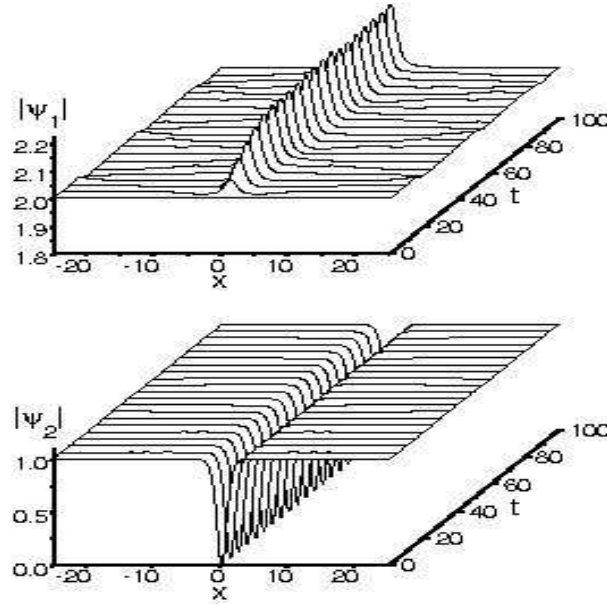
$$\begin{aligned} \delta\rho_1(r, t) &= \frac{1}{4\pi} \int_0^\infty \delta\rho_1(k) J_0(kr) \left[ \left( 1 + \frac{g_{11}\rho_{10} - g_{22}\rho_{20}}{c_+^2 - c_-^2} \right) \cos \left( tk\sqrt{c_+^2 + \frac{k^2}{4}} \right) \right. \\ &\quad \left. + \left( 1 - \frac{g_{11}\rho_{10} - g_{22}\rho_{20}}{c_+^2 - c_-^2} \right) \cos \left( tk\sqrt{c_-^2 + \frac{k^2}{4}} \right) \right] k dk \\ &\quad + \frac{g_{12}\rho_{10}}{2\pi(c_+^2 - c_-^2)} \int_0^\infty \delta\rho_2(k) J_0(kr) \left[ \cos \left( tk\sqrt{c_+^2 + \frac{k^2}{4}} \right) - \cos \left( tk\sqrt{c_-^2 + \frac{k^2}{4}} \right) \right] k dk, \end{aligned} \quad (47)$$

$$\begin{aligned} \delta\rho_2(r, t) &= \frac{1}{4\pi} \int_0^\infty \delta\rho_2(k) J_0(kr) \left[ \left( 1 - \frac{g_{11}\rho_{10} - g_{22}\rho_{20}}{c_+^2 - c_-^2} \right) \cos \left( tk\sqrt{c_+^2 + \frac{k^2}{4}} \right) \right. \\ &\quad \left. + \left( 1 + \frac{g_{11}\rho_{10} - g_{22}\rho_{20}}{c_+^2 - c_-^2} \right) \cos \left( tk\sqrt{c_-^2 + \frac{k^2}{4}} \right) \right] k dk \\ &\quad + \frac{g_{12}\rho_{20}}{2\pi(c_+^2 - c_-^2)} \int_0^\infty \delta\rho_1(k) J_0(kr) \left[ \cos \left( tk\sqrt{c_+^2 + \frac{k^2}{4}} \right) - \cos \left( tk\sqrt{c_-^2 + \frac{k^2}{4}} \right) \right] k dk. \end{aligned} \quad (48)$$

It is easy to see that if there is no interaction between two components of BECs, then these formulae reduce to the already known results for one-component condensates.

Integrals in (45)-(48) can be estimated by the method of stationary phase which gives the approximate expressions for disturbances of the densities in the regions  $x > c_+t$  or  $r > c_+t$  where we can observe the interference pattern of two modes.

The present setting with homogeneously mixed two repulsive BECs corresponds to the most relevant experimental situation. The case of two repulsive 1D BECs is akin to previously discussed one (see Fig. 1), if there is a dark soliton in one component. When the repulsive inter-species interaction is slowly turned on a density hump emerges in the other component, as illustrated in Fig. 5. Linear waves then can be generated by a fast turning off the inter-species interaction.

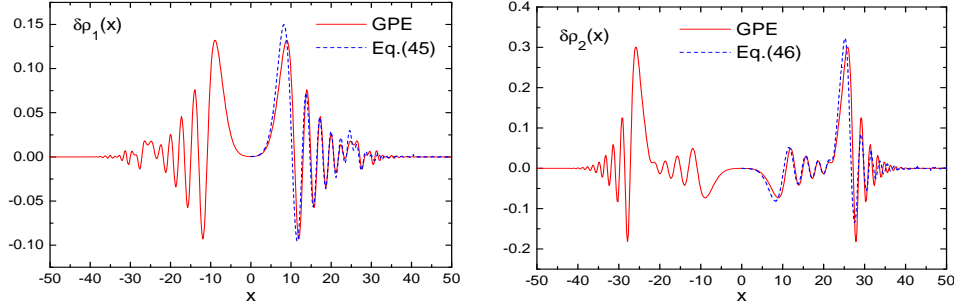


**Figure 5.** Formation of a density defect in two coupled 1D repulsive BEC's with initial wave amplitudes  $\psi_1(x) = 2$ ,  $\psi_2(x) = \tanh(x)$ , and  $g_{11} = 1$ ,  $g_{22} = 1$ , when the repulsive inter-species interaction is slowly turned on according to law  $g_{12} = 0.5 \tanh(5t/t_{end})$ , with  $t_{end} = 100$ . Dark soliton in one component gives rise to a density hump in the other component.

Another approach for generation of density disturbances in coupled repulsive BEC's involves local change of the intra- or inter-species interaction constants by a tightly focused laser beam via optically induced Feshbach resonance [22]. Stationary bright-bright, dark-dark and dark-bright coupled localized states on finite backgrounds can emerge in mixture BEC's when the interaction constants are locally changed. In particular, a stationary bright-bright (dark-dark) localized state in all repulsive case occurs if the inter-species repulsion is locally reduced (increased). A bright-dark localized state emerges when the intra-species repulsion is locally reduced in one of the components.

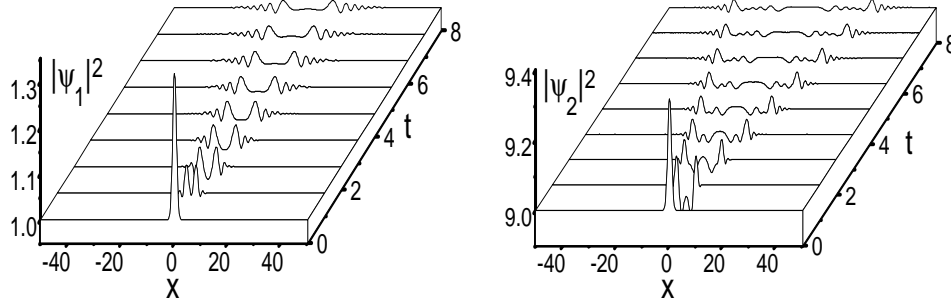
After the density defect (hump or hole) has been created, a fast change of the inter-species interaction coefficient leads to generation of waves in the condensate caused by decaying density disturbance. These results are illustrated in Fig. 6, where comparison

between the analytical prediction and numerical simulation of the GPE is also presented.



**Figure 6.** Density modulations at  $t = 8$  according to Eqs. (45)-(46) and numerical solution of the coupled GPE. Parameters:  $\rho_{10} = 1$ ,  $\rho_{20} = 9$ ,  $a = 1$ ,  $g_{11} = 1$ ,  $g_{22} = 1$ ,  $g_{12} = 0.5$ .

Since there is no stable localized solution (like dark soliton in 1D repulsive case) of the GP equation in higher dimensional settings, creation of density disturbance in this case also can be done by illuminating the homogeneous mixture BEC by tightly focused laser beam designed to change the intra- or inter-species interaction constant via optically induced Feshbach resonance.

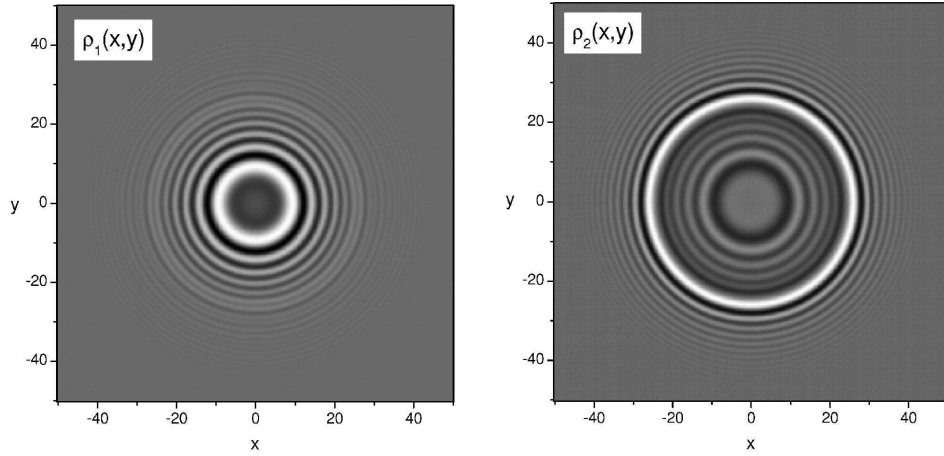


**Figure 7.** Evolution of waves in a 2D mixture BEC emerged from the decay of a Gaussian density disturbance with  $a = 1$  in both components shown at  $y = 0$  cross section according to numerical solution of the coupled GP equations. Background wave amplitudes of components and nonlinear coefficients are equal to  $\psi_1(x, y) = 1$ ,  $\psi_2(x, y) = 3$ ,  $g_{11} = 1$ ,  $g_{22} = 1$ ,  $g_{12} = 0.5$ . Due to different sound velocities in the components radially outward travelling waves quickly separate.

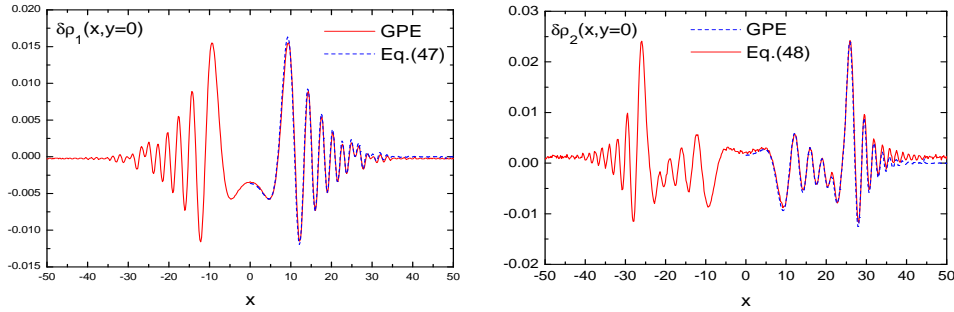
In Figs. 7–9 the generation, further evolution and pattern of linear waves at particular time are depicted for the 2D coupled repulsive BEC's. As can be seen from Figs. 4, 6 and 9, the agreement between analytical predictions and numerical solution of the GPE is quite good.

## 5. Conclusions

We have investigated sound waves in two-component BECs and proposed a new method of wave generation which is based on a fast change of the inter-species interaction



**Figure 8.** Density plots of two components corresponding to the last time section at  $t = 8$  of the previous figure 7. In white areas the density is higher. Separation of waves in the two components due to different sound velocities is clearly seen.



**Figure 9.** Comparison of density distributions provided by Eqs. (47) and (48) with numerical solution of the coupled GP equation for the same parameters as in Fig. 7 for  $t = 8$ . Relative positions of the wave maximums in the second and first modes  $26/9 \simeq 3$  well match the ratio of corresponding sound velocities  $c_+/c_-$  which have the same order of magnitude as group velocities of our wave packets built of harmonics with wavelengths about healing length.

constant. To illustrate the idea, we have performed numerical simulations of the creation of a density defect (hump or hole) on the background condensate and its evolution into matter wavepackets after the fast change of the interspecies interaction constant. This method of wave generation was used to investigate both numerically and analytically sounds waves arising from the release of a drop-like condensate immersed into a second large repulsive condensate and from homogeneous mixture of two repulsive BEC's both with finite backgrounds. We demonstrated that propagation of linear waves created by this approach is well described by the Gross-Pitaevskii equation. Explicit analytical formulae were obtained for the space-time evolution of density waves in coupled BECs, starting from the linearized GP equations. The comparison of the analytical results with direct numerical simulations of the coupled GPE equations showed an excellent agreement.

Although in this paper we have restricted our considerations to weak density

disturbances (e.g. amplitude of the disturbance is small compared to the amplitude of the background condensate), the proposed method can be used to investigate also ship waves and dispersive shocks in binary BEC mixtures as it will be discussed elsewhere.

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